J Glob Optim (2010) 48:465–471
DOI 10.1007/s10898-009-9500-5

Exceptional family of elements for generalized variational inequalities

Zhi Liu · **Yiran He**

Received: 29 July 2008 / Accepted: 26 November 2009 / Published online: 10 December 2009

Abstract This paper introduces a new concept of exceptional family of elements for a finite-dimensional generalized variational inequality problem. Based on the topological degree theory of set-valued mappings, an alternative theorem is obtained which says that the generalized variational inequality has either a solution or an exceptional family of elements. As an application, we present a sufficient condition to ensure the existence of a solution to the variational inequality. The set-valued mapping is assumed to be upper semicontinuous with nonempty compact convex values.

Keywords Generalized variational inequality · Topological degree · Exceptional family of elements · Coercivity condition

1 Introduction

Throughout this paper, let R^n be the n-dimensional Euclidean space, $K \subseteq R^n$ be a non-empty closed convex set, and $F : K \to 2^{R^n}$ be a set-valued mapping. The variational inequality problem, denoted by VI(K, F), is to find $x \in K$ and $x^* \in F(x)$ such that

$$\langle x^*, y - x\rangle \geq 0, \quad \text{for all } y \in K. \tag{1.1}$$

When K is a closed convex cone, the above problem reduces to a complementary problem, which is to find $x \in K$ and $x^* \in F(x)$ such that

$$x^* \in K^+ \text{ and } \langle x^*, x\rangle = 0, \text{ where } K^+ \equiv \{d \in R^n : \langle v, d\rangle \geq 0, \forall v \in K\}. \tag{1.2}$$

The authors are grateful to the referees for valuable suggestions. This work is partially supported by National Natural Science Foundation of China (No. 10701059) and by Sichuan Youth Science and Technology Foundation (No. 06ZQ026-013).

Z. Liu · Y. He (✉)
Department of Mathematics, Sichuan Normal University, 610066 Chengdu, Sichuan, China
e-mail: yiranhe@hotmail.com

Variational inequality and complementarity problems have been extensively studied in the literature; see [4–6,20]. In the case that the mapping F is single-valued, the method of exceptional family is used to deal with the existence of solutions to variational inequality problems and complementary problems (see [2,7–16,19,21–25]). These references introduced a concept of exceptional family of elements and applied it to prove the existence of solutions to the variational inequality where the mapping F is assumed to be single-valued and continuous. However, it remains unknown whether the exceptional family method can be used to deal with the generalized variational inequalities where F is a set-valued mapping. In this paper, we introduce a concept of exceptional family for generalized variational inequalities and prove that if F is upper semicontinuous with nonempty compact convex values, then VI(K, F) has either a solution or an exceptional family of elements. We also present a coercivity condition to ensure the existence of solutions to VI(K, F). Our results extend the main results in [7] and the references therein.

2 Preliminary results

Without other specifications, let $K \subseteq R^n$ be a non-empty closed convex set, $\Omega \subset R^n$ be an open bounded set and $K \cap \Omega \neq \emptyset$, $F : K \to 2^{R^n}$ be a set-valued mapping. F is said to be upper semicontinuous at $x \in K$ if for any open set $V \subset R^n$ such that $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V$ for all $y \in U \cap K$; If F is upper semicontinuous at every $x \in K$, we say F is upper semicontinuous on K.

In this paper the common symbols are as follows: the closure and boundary of Ω are denoted by $\overline{\Omega}$ and $\partial\Omega$, respectively. The Euclidean norm is denoted by $\|\cdot\|$, the projection operator on K is denoted by $P_K(\cdot)$. The normal cone of K at x is denoted by $N_K(x)$, that is,

$$N_K(x) = \begin{cases} \{x^* \in R^n : \langle x^*, y - x\rangle \leq 0, \forall y \in K\}, & \text{if } x \in K. \\ \emptyset, & \text{otherwise .} \end{cases}$$

Recently, Kien et al. [17] built a degree theory for generalized variational inequalities. Let $F : K \to 2^{R^n}$ be upper semicontinuous set-valued mapping with compact convex values and $0 \notin (F + N_K)(\partial\Omega)$. The degree of generalized variational inequality defined by F and K respect to Ω at 0 is the common value $d(\Phi_\epsilon, \Omega, 0)$ for $\epsilon > 0$ sufficiently small and denoted by $d(F + N_K, \Omega, 0)$, where $\Phi_\epsilon(x) = x - P_K(x - f_\epsilon(x))$ and f_ϵ is a approximate continuous selection of F (see [17, Lemma 2.1]). To show our main results, we need the following properties.

Lemma 2.1 (Existence) *Suppose that $F : K \to 2^{R^n}$ is upper semicontinuous set-valued mapping with compact convex values and $0 \notin (F + N_K)(\partial\Omega)$. If $d(F + N_K, \Omega, 0) \neq 0$, then there exists $x \in \Omega \cap K$ such that*

$$0 \in F(x) + N_K(x). \tag{2.1}$$

Proof See Theorem 2.1 in [17]. □

Remark 2.1 It is clear that x is a solution of (2.1) if and only if x is a solution of (1.1). Since $0 \in F(x) + N_K(x)$, then $-F(x) \subset N_K(x)$. By the definition of $N_K(x)$, we know that there exists $x^* \in F(x)$ such that $\langle x^*, y - x\rangle \geq 0$, for all $y \in K$.

Proposition 2.1 (Normalization) *Suppose that $F(x) = x - \widehat{x}$ and $0 \notin (x - \widehat{x} + N_K)(\partial\Omega)$, where $\widehat{x} \in R^n$. If $0 \in \Omega$, then $d(x - \widehat{x} + N_K(x), \Omega, 0) = 1$.*

Proof Since $F(x) = x - \widehat{x}$, then F is upper semicontinuous with compact convex values on K. So we can use the definition of topological degree, then

$$d(x - \widehat{x} + N_K(x), \Omega, 0) = d(\Phi_\epsilon, \Omega, 0), \text{ where } \Phi_\epsilon(x) = x - P_K(x - f_\epsilon(x)).$$

Let $f_\epsilon(x) = x - \widehat{x}$, then

$$d(x - \widehat{x} + N_K(x), \Omega, 0) = d(x - \overline{x}, \Omega, 0), \text{ where } \overline{x} = P_K(\widehat{x}).$$

Since the mapping $x - \overline{x}$ is a translation of the identity mapping x, then $d(x - \overline{x}, \Omega, 0) = d(x, \Omega + \overline{x}, \overline{x}) = 1$. So $d(x - \widehat{x} + N_K(x), \Omega, 0) = 1$. □

Lemma 2.2 (Homotopy invariance) *Let* $F_1, F_2 : K \to 2^{R^n}$ *be upper semicontinuous set-valued mapping with compact convex values and* $0 \notin (tF_1 + (1-t)F_2 + N_K)(\partial\Omega)$ *for all* $t \in [0, 1]$*, then*

$$d(F_1 + N_K, \Omega, 0) = d(F_2 + N_K, \Omega, 0).$$

Our definition of an exceptional family of elements is as follows:

Definition 2.1 Let $\widehat{x} \in R^n$. A family of elements $\{x_r\}_{r>0} \subset K$ is said to be an exceptional family for VI(K, F) with respect to $\widehat{x}$ if:

(i) $\|x_r\| \to \infty$ as $r \to \infty$.
(ii) for any $r > \|P_K(\widehat{x})\|$, there exist a real number $\alpha_r > 0$ and $y_r \in F(x_r)$ such that $-y_r + \alpha_r(\widehat{x} - x_r) \in N_K(x_r)$,
where $N_K(x_r) := \begin{cases} \{y \in R^n : \langle y, z - x_r\rangle \le 0, \forall z \in K\}, & \text{if } x_r \in K. \\ \emptyset, & \text{if } x_r \in R^n \backslash K. \end{cases}$

Remark 2.2 If the mapping F is single-valued, Definition 2.1 reduces to Definition 2.1 in [7]. Since $N_K(x_r)$ is a cone, it can be seen that Definition 2.1 coincides with Definition 5.1 in [10], by letting $\hat{x} = 0$ and $\alpha_r := \frac{1}{t_r} - 1$ where t_r appears in Definition 5.1 in [10].

When F is a set-valued mapping, we compare Definition 2.1 and Definition 4.2 in [15]. If the set K is a closed convex cone and $\hat{x} = 0$, then Definition 4.1 in [15] coincides with Definition 2.1. However, if K is not a cone, there is a difference between Definition 4.2 in [15] and Definition 2.1: the former used $N_K((\alpha_r + 1)x_r)$ to replace $N_K(x_r)$. This difference is also observed by Professor Isac; see Remark 3.3 in [10].

In many applications of variational inequalities the set K is described by

$$K := \{x \in R^n : g_i(x) \le 0, i = 1, \dots, m;\ h_j(x) = 0, j = 1, \dots, l\}, \tag{2.2}$$

where $g_i : R^n \to R$ is a continuously differentiable convex function and $h_j : R^n \to R$ is an affine function, and there exists $x_0 \in K$ such that $g_i(x_0) < 0$ for $i = 1, \dots, m$ and $h_j(x_0) = 0$ for $j = 1, \dots, l$. In this case we introduce the following notion of exceptional family of elements.

Definition 2.2 Let K be a nonempty closed convex set defined by (2.2) and $\widehat{x} \in R^n$. A family of elements $\{x_r\}_{r>0} \subset K$ is said to be an exceptional family for VI(K, F) with respect to $\widehat{x}$ if:

(i) $\|x_r\| \to \infty$ as $r \to \infty$.
(ii) for each $r > 0$, there exist $\lambda_r \in R^m$, $\mu_r \in R^l$, $\alpha_r \in R^1$, $y_r \in F(x_r)$ such that

$$y_r \in -\alpha_r(x_r - \widehat{x}) - (\nabla g(x_r)^T \lambda_r + \nabla h(x_r)^T \mu_r),$$
$$\lambda_r^T g(x_r) = 0, \lambda_r \geq 0, \alpha_r > 0,$$

where $\nabla g(x) \in R^{m\times n}$ and $\nabla h(x) \in R^{l\times n}$ are the Jacobian matrices of g and h, respectively.

Remark 2.3 When K is defined by (2.2), we know that $N_K(x_r) = \{\nabla g(x_r)^T \lambda_r + \nabla h(x_r)^T \mu_r : \lambda_r \in R^m, \mu_r \in R^l, \lambda_r^T g(x_r) = 0\}$. So Definition 2.1 is equivalent to Definition 2.2 in this case. Moreover, if F is a single-valued continuous mapping and $\widehat{x} = 0$, (i) and (ii) in Definition 2.2 provide exactly the same definition of the exceptional family as in [25].

3 Main results

Theorem 3.1 *If $F : K \to 2^{R^n}$ is an upper semicontinuous set-valued mapping with nonempty compact convex values. Then either VI(K, F) has a solution or, for every point $\widehat{x} \in R^n$, there exists an exceptional family for VI(K, F) with respect to $\widehat{x}$.*

Proof By Remark 2.1, we know that the solvability of VI(K, F) is equivalent to the equation $0 \in (F + N_K)(x)$ being solvable in K. For any $\widehat{x} \in R^n$, we consider the homotopy between the mappings $x - \widehat{x} + N_K(x)$ and $(F + N_K)(x)$, which is defined by

$$H(x, t) = tF(x) + (1 - t)(x - \widehat{x}) + N_K(x), \quad \forall t \in [0, 1].$$

Let $D_r = \{x \in R^n : \|x\| < r\}$, which is such that $D_r \cap K \neq \emptyset$. Then we have the the following results: If for some $r > \|P_K(\widehat{x})\|$, $0 \in (F + N_K)(\partial D_r)$, then VI($K$, F) has a solution. Now assume that for any $r > \|P_K(\widehat{x})\|$, $0 \notin (F + N_K)(\partial D_r)$, the following two cases are possible.

(a) There exists an $r_0 > \|P_K(\widehat{x})\|$ such that $0 \notin H(\partial D_{r_0}, [0, 1])$, thus the homotopy invariance implies that

$$d(H(x, 1), D_{r_0}, 0) = d(H(x, 0), D_{r_0}, 0),$$

where $H(x, 0) = x - \widehat{x} + N_K(x)$, $H(x, 1) = (F + N_K)(x)$.
By Proposition 2.1, then $d(H(x, 1), D_{r_0}, 0) = d(H(x, 0), D_{r_0}, 0) = 1$. So the existence implies that the ball $\overline{D}_{r_0}$ contains at least one solution to the equation $0 \in (F + N_K)(x)$. Therefore VI(K, F) has a solution.

(b) For each $r > \|P_K(\widehat{x})\|$, there exist a point $x_r \in \partial D_r$ and a scalar $t_r \in [0, 1]$ such that $0 \in H(x_r, t_r)$.

We now claim that $t_r \neq 0$ and $t_r \neq 1$. If $t_r = 0$, then $\widehat{x} - x_r \in N_K(x_r)$. It follows that $x_r = P_K(\widehat{x})$, which is impossible since $\|x_r\| = r$ and $r > \|P_K(\widehat{x})\|$. If $t_r = 1$, then $0 \in (F + N_K)(\partial D_r)$, which is also impossible since $0 \notin (F + N_K)(\partial D_r)$. Therefore, for each $r > \|P_K(\widehat{x})\|$, there exist a point $x_r \in \partial D_r$ and a scalar $t_r \in (0, 1)$ such that $0 \in H(x_r, t_r)$, that is,

$$0 \in t_r F(x_r) + (1 - t_r)(x_r - \widehat{x}) + N_K(x_r).$$

Thus there exists a $y_r \in F(x_r)$ such that $0 \in t_r y_r + (1-t_r)(x_r - \widehat{x}) + N_K(x_r)$
or

$$\frac{(\widehat{x} - x_r)(1-t_r)}{t_r} - y_r \in N_K(x_r).$$

Let $\alpha_r = (1-t_r)/t_r > 0$, then $-y_r + \alpha_r(\widehat{x} - x_r) \in N_K(x_r)$. On the other hand, since $\|x_r\| = r$, $\|x_r\| \to \infty$ as $r \to \infty$. Moreover, when $0 < r \leq \|P_K(\widehat{x})\|$, we may choose x_r to be any point in K. Hence, we can find an exceptional family $\{x_r\}_{r>0}$ for VI(K, F) with respect to $\widehat{x}$. □

Remark 3.1 When F is single-valued, [7, p. 511] presented an example of variational inequality which has no exceptional family in the sense of Definition 2.1 (and so has a solution), but does have an exceptional family in the sense of Definition 4.2 in [15].

In order to show the existence of solutions to variational inequality problems, various coercivity conditions have been used (see [1–4]). This paper picks out two of these:

(C_1) There exists a nonempty bounded subset D of K such that, for every $x \in K\backslash D$, there is a $y \in D$ satisfying $\inf_{y^* \in F(x)} \langle y^*, x - y\rangle \geq 0$.

(C_2) There exists a constant $r > 0$ such that, for any $x \in K$ with $\|x - \widehat{x}\| > r$, where $\widehat{x} \in R^n$, then there exists $y \in K$ satisfying $\|y - \widehat{x}\| < \|x - \widehat{x}\|$ and $\inf_{y^* \in F(x)} \langle y^*, x - y\rangle \geq 0$.

Obviously, $C_1 \Rightarrow C_2$, since $D \subset K$ is bounded, there must exist a enough large real number $r > 0$ such that $D \subset B_r \cap K$, where $B_r = \{x \in R^n : \|x - \widehat{x}\| \leq r\}$. Let $x \in K\backslash D$. Therefore the condition (C_1)implies that, for every $x \in K\backslash D$, there is a $y \in D$ satisfying $\|y - \widehat{x}\| \leq r < \|x - \widehat{x}\|$ and $\inf_{y^* \in F(x)} \langle y^*, x - y\rangle \geq 0$.

From the above, condition (C_2) is a rather weak coercivity among most of known coercivity conditions. Specially, in [2], this condition was used to prove that VI(K, F) has a solution when F is qusimonotone upper sign-continuous set-valued mapping with nonempty weakly compact values in Banach space. We will prove the existence theorem when F is an upper semicontinuous set-valued mapping with nonempty compact convex in R^n.

Theorem 3.2 *If $F : K \to 2^{R^n}$ is an upper semicontinuous set-valued mapping with nonempty compact convex values and the condition (C_2) holds, then VI(K, F) has no exceptional family with respect to $\widehat{x}$; hence VI(K, F) has a solution.*

Proof Assume that VI(K, F) has no solution. By Theorem 3.1, we know that there exist an exceptional family of elements $\{x_r\}_{r>0}$ for VI(K, F) with respect to $\widehat{x}$, that is, $\{x_r\}_{r>0} \subset K$ with $\|x_r\| \to \infty$, as $r \to \infty$, and for any $r > \|P_K(\widehat{x})\|$, there exist a real number $\alpha_r > 0$ and $y_r \in F(x_r)$ such that $-y_r + \alpha_r(\widehat{x} - x_r) \in N_K(x_r)$, that is,

$$\langle y_r + \alpha_r(x_r - \widehat{x}), y - x_r\rangle \geq 0, \quad \forall y \in K. \tag{3.1}$$

Since that the condition (C_2) holds, then there exists a constant $r > 0$ such that, for any $x_r \in K$ with $\|x_r - \widehat{x}\| > r$, there exists $y \in K$ satisfying $\|y - \widehat{x}\| < \|x_r - \widehat{x}\|$ and $\inf_{y_r \in F(x_r)} \langle y_r, x_r - y\rangle \geq 0$. So we can claim that

$$\langle y_r, y - x_r\rangle \leq 0, \quad \forall y_r \in F(x_r). \tag{3.2}$$

Meanwhile, since $\langle x_r - \widehat{x}, y - \widehat{x}\rangle \leq \|x_r - \widehat{x}\|\|y - \widehat{x}\|$, then

$$\alpha_r\langle x_r - \widehat{x}, y - \widehat{x}\rangle \leq \alpha_r\|x_r - \widehat{x}\|\|y - \widehat{x}\|, \quad \forall \alpha_r > 0. \tag{3.3}$$

Adding the formulas (3.2) and (3.3), then for $\forall y_r \in F(x_r)$ and $\forall \alpha_r > 0$,

$$\langle y_r, y - x_r\rangle + \alpha_r\langle x_r - \widehat{x}, y - \widehat{x}\rangle \leq \alpha_r\|x_r - \widehat{x}\|\|y - \widehat{x}\|.$$

It follows that for $\forall y_r \in F(x_r)$ and $\forall \alpha_r > 0$,

$$\begin{aligned}&\langle y_r + \alpha_r(x_r - \widehat{x}), y - x_r\rangle\\ &\quad = \langle y_r, y - x_r\rangle + \alpha_r\langle x_r - \widehat{x}, y - \widehat{x} + \widehat{x} - x_r\rangle\\ &\quad = \langle y_r, y - x_r\rangle + \alpha_r\langle x_r - \widehat{x}, y - \widehat{x}\rangle - \alpha_r\|x_r - \widehat{x}\|^2\\ &\quad \le \alpha_r\|x_r - \widehat{x}\|(\|y - \widehat{x}\| - \|x_r - \widehat{x}\|)\\ &\quad < 0. \end{aligned} \tag{3.4}$$

It is clear that (3.1) in contradiction with (3.4). Thus, VI(K, F) has no exceptional family with respect to $\widehat{x}$; hence, VI(K, F) has a solution. □

Remark 3.2 Theorem 3.2 can also be obtained from Theorem 2.5 in [18], a very general result stated for equilibrium problem.

Corollary 3.1 *If $F : K \to 2^{R^n}$ is an upper semicontinuous set-valued mapping with nonempty compact convex values and the condition (C_1) holds, then for VI(K, F) has no exceptional family with respect to $\widehat{x}$, where $\widehat{x} \in R^n$; hence VI(K, F) has a solution.*

Proof Since $C_1 \Rightarrow C_2$, it follows from Theorem 3.2 that VI(K, F) has a solution. □

References

1. Aussel, D., Hadjisavvas, N.: On quasimonotone variational inequalities. J. Optim. Theory Appl. **121**, 445–450 (2004)
2. Bianchi, M., Hadjisavvas, N., Schaible, S.: Minimal coercivity conditions and exceptional families of elements in quasimonotone varaiational in equalities. J. Optim. Theory Appl. **122**, 1–17 (2004)
3. Daniilidis, A., Hadjisavvas, N.: Coercivity conditions and variational inequalities. Math. Program. **86**, 433–438 (1999)
4. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, New York (2003)
5. Giannessi, F., Maugeri, A., Pardalos, P.M. (eds.): Equilibrium Problems and Variational Models. Kluwer, Dordrecht (2001)
6. Gowda, M.S.: Application of degree theory to linear complementarity problems. Math. Oper. Res. **18**(4), 868–879 (1993)
7. Han, J., Huang, Z.H., Fang, S.C.: Solvability of variational inequality problems. J. Optim. Theory Appl. **122**, 501–520 (2004)
8. Huang, N.J., Gao, C.J.: Exceptional family of elements and feasibility for nonlinear complementarity problems. J. Glob. Optim. **25**, 337–344 (2003)
9. Isac, G.: A generalization of Karamardian's condition in complementarity theory. Nonlinear Anal. Forum **4**, 49–63 (1999)
10. Isac, G.: Complementarity problems and variational inequalities. A unified approach of solvability by and implicit Leray-Schauder type alternative. J. Glob. Optim. **31**, 405–420 (2005)
11. Isac, G.: Leray-Schauder Type Alternative. Complementarity Problems and Variational Inequalities. Springer, New York (2006)
12. Isac, G., Carbone, A.: Exceptional family of elements for continuous functions: some application to complementarity theory. J. Glob. Optim. **15**, 191–196 (1999)
13. Isac, G., Kalashnikov, V.V.: Exceptional families of elements, Leray-Schauder alternative, pseudomonotone operators and complementarity. J. Optim. Theory Appl. **109**(1), 2345–2349 (2001)
14. Isac, G., Obuchowska, W.T.: Functions without exceptional families of elments and complementarity problem. J. Optim. Theory Appl. **99**, 147–163 (1998)
15. Isac, G., Zhao, Y.B.: Exceptional family of elements and solvability of variational inequalities for unbounded sets in infinite dimensional Hilbert spaces. J. Math. Anal. Appl. **246**, 544–556 (2000)
16. Isac, G., Bulavski, V., Kalashnikov, V.: Exceptional families, topological degree and complementarity. J. Glob. Optim. **10**, 207–225 (1997)

17. Kien, B.T., Wong, M.-M., Wong, N.C., Yao, J.C.: Degree theory for generalized variational inequalities and applications. Eur. J. Oper. Res. **192**(3), 730–736 (2009)
18. Konnov, I.: Regularization method for nonmonotone equilibrium problems. J. Nonl. Conv. Anal. **10**, 93–101 (2009)
19. Li, J.L., Whitaker, J.: Exceptional family of elements and solvability of varational inequalities for mappings defined only on closed convex cones in Banach spaces. J. Math. Anal. Appl. **310**, 254–261 (2005)
20. Pardalos, P.M., Rassias, T.M., Khan, A.A. (eds.): Nonlinear Analysis and Variational Problems, In Honor of George Isac. Springer, New York (2001)
21. Zhang, L., Han, J., Hu, D.: Existence theorems of solution to variational inequality problems. Sci. China Ser. A **44**, 212–219 (2001)
22. Zhao, Y.B.: Exceptional families and finite-dimensional variational inequalities over polyhedral convex sets. Appl. Math. Comput. **87**, 111–126 (1997)
23. Zhao, Y.B.: Existence of a solution to nonlinear variational inequality under generalized positive homogeneity. Oper. Res. Lett. **25**, 231–239 (1999)
24. Zhao, Y.B., Han, J.: Exceptional family of elements for a variational inequality problem and its applications. J. Glob. Optim. **14**, 313–330 (1999a)
25. Zhao, Y.B., Han, J.: Exceptional families and existence theorems for variational inequality problems. J. Optim. Theory Appl. **101**, 475–495 (1999b)